Proceedings of the ASME 2018 Pressure Vessels and Piping Conference
PVP2018
July 15-20, 2018, Prague, Czech Republic

PVP2018-84784

Application of An *a priori* Jacobian-based Error Estimation Metric to the Accuracy Assessment of 3D Finite Element Simulations (*)

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ABSTRACT

The determinant of the Jacobian matrix is frequently used in the Finite Element Method as a measure of mesh quality.

A new metric is defined, called the Standard Error, based on the distribution of the determinants of the Jacobian matrices of all elements of a finite element mesh. Where the Jacobian norm can be used to compare the quality of one element to another of the same type, the Standard Error compares the mesh quality of different versions of a finite element model where each version uses a different element type.

To motivate this new Standard Error, we investigate the geometric meaning of the Jacobian norm on 3D Finite Elements. This mesh quality metric is applied to 8, 20, and 27 node hexahedra, 6 and 15 node prisms, 4 and 10 node tetrahedra, 5 and 13 node pyramid, and 3, 4, 6, 8, and 9 node shell elements. The shape functions for these 14 element types, or more precisely their first partial derivatives, are used to construct the Jacobian Matrix. The matrix is normalized to compensate for size. The determinant of the Jacobian is calculated at Gaussian points within each element. Statistics are gathered to form the Standard Error of the mesh.

To illustrate the applicability of this *a priori* metric, we present two simple example problems having exact answers, and two industry-type problems, a pipe elbow with a crack and a magnetic resonance imaging (MRI) birdcage RF coil resonance, both having no analytical solution.

Significance and limitations of using this *a priori* metric to assess the accuracy of finite element simulations of different mesh designs are presented and discussed.

1. INTRODUCTION

An essential component for verification and validation of computer simulations of high-consequence engineering systems is a parametric FEA pre-processor with the following requirements:

(1) Element type: It must be able to choose the element type to be generated from 8, 20, and 27 node hexahedra, 6 and 15 node prisms, 4 and 10 node tetrahedra, 5 and 13 node pyramid, and 3, 4, 6, 8, and 9 node shell elements.
(2) Mesh Density: The mesh density can be changed locally or globally by changing a few parameters and preferably just one.
(3) Model Parameters: Parameters and algebraic forms can be used to change geometry, mesh topology, boundary conditions, constraints, loads, and materials by changing the values of a few parameters.
(4) Solution Platform: The model generated using this pre-processor can be translated to any of a set of Finite Element simulation codes including ABAQUS, ANSYS, COMSOL, LS-DYNA, MPACT, and NASTRAN.

The commercially available pre-processor called **True*Grid*®**, a super-parametric mesh generator [5][10], meets

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these requirements and is used throughout this paper to generate various elements and to test their quality.

Numerous parametric FEA models have been generated to demonstrate numerical methods of verification and validation of the computer simulations. This is done by varying the mesh density, element type, and simulation platform. When things go well, there is an estimated asymptotic point of convergence. This is referred to *a posteriori* test of the simulation.

The goal of this paper is to determine if the mesh quality tests available in the pre-processor, referred to as *a priori* tests of the model, can predict the accuracy of the simulation. So a new quality test has been added to the pre-processor, called Standard Error, that produces a single number that can be used to compare all models of the same engineering system. The two important properties of this test are that it treats all element types equally and it is not influenced by mesh density.

This new test for quality is based on the Jacobian matrix. So the next section of this paper covers the fundamentals of the Jacobian matrix and its derivative. The shape functions for the various element types are included. The size of the element is factored out of the determinant of the Jacobian Matrix. The result is called the Jacobian Metric.

The next section has examples of elements and their Jabcobian measures. This is followed by the definition of the Standard Error, 4 examples, conclusions, references, and an appendix.

2. Shape Functions and Determinants

This paper considers fourteen 3D elements: 8, 20, and 27 node hexahedra, 6 and 15 node prisms, 4 and 10 node tetrahedra, 5 and 13 node pyramids, and 3, 4, 6, 8, and 9 node shells. This paper also considers their shape functions and their corresponding Jacobian determinants. All of these elements can be defined as a 3D function. The domain of this function is sometimes referred to as the space of three normalized parametric coordinates. We will use the parameter names α, β, and γ. The allowed values of these parameters will vary depending on the element type.

Each element is defined by a set of ordered nodes with coordinates $p_i = (x_i, y_i, z_i)$ for i between 1 and m, where m is the number of nodes in the element. Associated with each node is a scalar shape function $N_i(\alpha,\beta,\gamma)$. We define the function $f(\alpha,\beta,\gamma)$ that maps the domain of normalized coordinates to the element [1].

$$f(\alpha,\beta,\gamma) = \sum_{1}^{m} N_i\,(\alpha,\beta,\gamma) * p_i$$

Note that both f and p_i are 3D vectors. In some cases, it will be more convenient to refer to the equivalent component functions that form the 3-tuple coordinates.

$$f(\alpha,\beta,\gamma) = (x(\alpha,\beta,\gamma), y(\alpha,\beta,\gamma), z(\alpha,\beta,\gamma))$$

where

$$x(\alpha,\beta,\gamma) = \sum_{1}^{m} N_i\,(\alpha,\beta,\gamma) * x_i$$

$$y(\alpha,\beta,\gamma) = \sum_{1}^{m} N_i\,(\alpha,\beta,\gamma) * y_i$$

$$z(\alpha,\beta,\gamma) = \sum_{1}^{m} N_i\,(\alpha,\beta,\gamma) * z_i$$

This will simplify the notation for the Jacobian matrix. Our first example is the 8 node hexahedral element (m=8) where all three parameters α, β, and γ are between -1 and 1 and with the shape functions:

$$\begin{aligned}
N_1 &= (1-\alpha)(1-\beta)(1-\gamma)/8\\
N_2 &= (1+\alpha)(1-\beta)(1-\gamma)/8\\
N_3 &= (1+\alpha)(1+\beta)(1-\gamma)/8\\
N_4 &= (1-\alpha)(1+\beta)(1-\gamma)/8\\
N_5 &= (1-\alpha)(1-\beta)(1+\gamma)/8\\
N_6 &= (1+\alpha)(1-\beta)(1+\gamma)/8\\
N_7 &= (1+\alpha)(1+\beta)(1+\gamma)/8\\
N_8 &= (1-\alpha)(1+\beta)(1+\gamma)/8
\end{aligned}$$

The first partial derivatives of the function $f(\alpha,\beta,\gamma)$ can be expressed as:

$$\frac{\partial f}{\partial \alpha}(\alpha,\beta,\gamma) = \sum_{1}^{m} \frac{\partial N_i}{\partial \alpha}(\alpha,\beta,\gamma) * p_i$$

$$\frac{\partial f}{\partial \beta}(\alpha,\beta,\gamma) = \sum_{1}^{m} \frac{\partial N_i}{\partial \beta}(\alpha,\beta,\gamma) * p_i$$

$$\frac{\partial f}{\partial \gamma}(\alpha,\beta,\gamma) = \sum_{1}^{m} \frac{\partial N_i}{\partial \gamma}(\alpha,\beta,\gamma) * p_i$$

All three partial derivatives above are 3D vectors which can be expanded to form the equivalent component derivatives. Each forms a column in the Jacobian matrix below [3].

$$\frac{\partial(x,y,z)}{\partial(\alpha,\beta,\gamma)} = \begin{bmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} & \frac{\partial x}{\partial \gamma} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} & \frac{\partial y}{\partial \gamma} \\ \frac{\partial z}{\partial \alpha} & \frac{\partial z}{\partial \beta} & \frac{\partial z}{\partial \gamma} \end{bmatrix}$$

Each of these nine partial derivatives in the matrix above can be expressed in terms of the partial derivatives of the shape functions. For example,

$$\frac{\partial x}{\partial \alpha} = \sum_{1}^{m} \frac{\partial N_i}{\partial \alpha} * p_i$$

where

$$\frac{\partial N_1}{\partial \alpha} = -(1-\beta)(1-\gamma)/8$$

$$\frac{\partial N_2}{\partial \alpha} = \quad (1-\beta)(1-\gamma)/8$$

$$\frac{\partial N_3}{\partial \alpha} = \quad (1+\beta)(1-\gamma)/8$$

$$\frac{\partial N_4}{\partial \alpha} = -(1+\beta)(1-\gamma)/8$$

$$\frac{\partial N_5}{\partial \alpha} = -(1-\beta)(1+\gamma)/8$$

$$\frac{\partial N_6}{\partial \alpha} = \quad (1-\beta)(1+\gamma)/8$$

$$\frac{\partial N_7}{\partial \alpha} = \quad (1+\beta)(1+\gamma)/8$$

$$\frac{\partial N_8}{\partial \alpha} = -(1+\beta)(1+\gamma)/8$$

A full list of the partial derivatives of these shape functions can be found in the appendix.

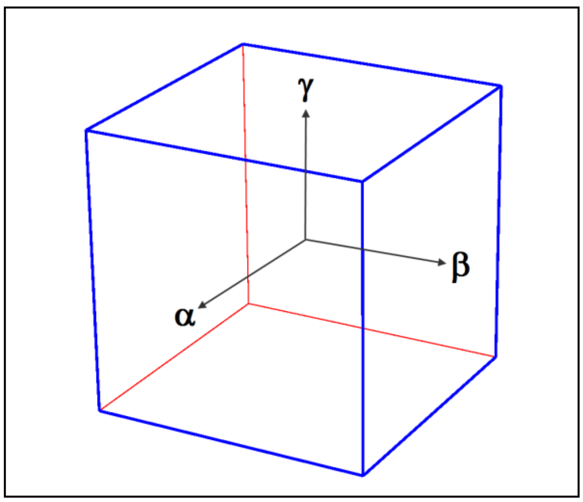

Fig. 1 Hexahedron Shape Function Domain

Figure 1 represents the domain of the function *f*. The red lines are the edges of the cube which are not usually visible. The corner where they meet has the parametric coordinates (-1,-1,-1). If you substitute these parametric values into the 8 shape functions, you will get 1 for N_1 and 0 for the others. It is this point in the domain that maps to the node at p_1 in the figure below. It is left to the reader to establish the correspondence between the remaining corners in the domain and the nodes (corners) of the element in the figure below.

It is required that the first four nodes have a positive normal, using the right hand rule, that points toward the center of the element while the last four nodes have a positive normal that points away from the center of the element. There are 24 valid ways to order these nodes. The Jacobian matrix for each of the 24 mappings may be different, but all the determinants of the Jacobians will be the same.

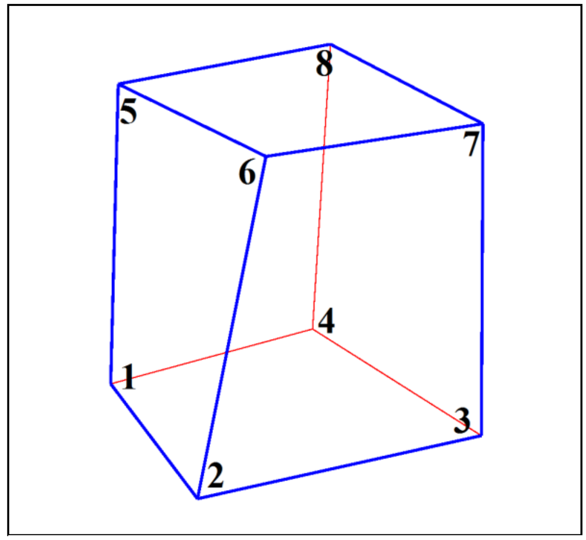

Fig. 2 Hexahedron Nodal Order

The 20 and 27 node hexahedral elements are similar in nature. The node numbering, the shape functions, and their partial derivatives can be found in the appendix.

When the mid-edge nodes of the quadratic elements are located at the mid-point of the line segment between 2 corner nodes, then the Jacobian Matrix is the same as the simpler first order elements. Examples of the Jacobian Metric applied to quadratic elements are omitted.

The 4 node tetrahedron is defined with parametric coordinates ranging from 0 to 1 with the added constraint that

$$\alpha + \beta + \gamma \le 1$$

The four shape functions are:

$$N_1 = \alpha$$
$$N_2 = \beta$$
$$N_3 = \gamma$$
$$N_4 = 1 - \alpha - \beta - \gamma$$

The partial derivatives of these shape functions are trivial to calculate and lead to an unusual property of the tetra. The determinant of its Jacobian matrix is a constant throughout the element. It does not matter how many Gaussian quadrature points used to sample the Jacobian measure, they will all be the same. This is not the case with higher order tetrahedrons.

The figure below represents the domain of the function *f* for the tetrahedron. The figure below shows one node ordering. The positive normal, using the right hand rule, to the plane through the first 3 nodes must point towards the 4th node. There are 12 valid ways to order the nodes in a tetrahedron. The determinant of the Jacobian matrix can be different for each nodal ordering. Below is one example of a nodal ordering.

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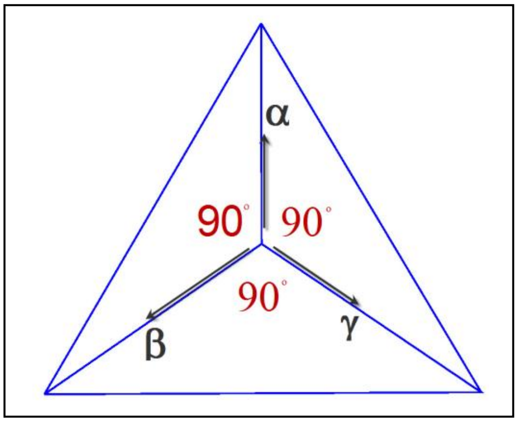


Fig. 3 Tetrahedron Shape Function Domain

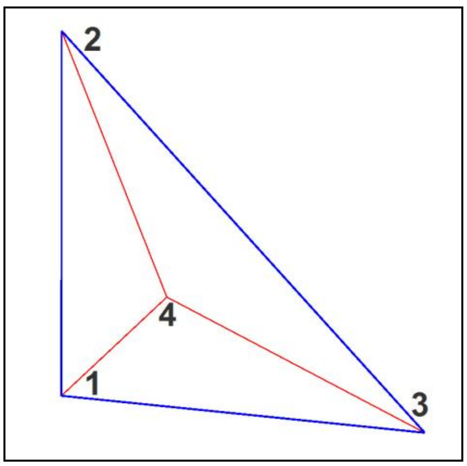


Fig. 4 Tetrahedron Nodal Numbering

The quadratic version of the tetrahedron has additional nodes located midway along each edge producing a total of 10 nodes. Full details on the 10 node tetrahedron are found in the appendix.

3. Jacobian Metric: Factoring Out Volume

Our function f can be viewed as a mapping from one 3D space to another. The Jacobian matrix approximates f with a matrix. And the determinant of that matrix tells us the scale factor change in volume. When we measure the quality of an element, we are not interested in the effect that the size of the element has on the determinant. To remove the size factor from the determinant of the Jacobian, we determine the singular values of the Jacobian. If J is our Jacobian matrix, we use the factorization known as the Singular Value decomposition [2]

$$J = U * D * V$$

where U and V are unitary matrices, and D is a diagonal matrix. The diagonal values of D are positive real numbers known as the singular values of J. If σ is the middle singular value, we define the ***Jacobian metric*** to be:

$$\frac{|J|}{\sigma^3}$$

3. Examples of the Jacobian Measure

The ideal hex element is a cube which has a Jacobian measure of 1.0. The core of a proof of this statement would be to consider any volume in the domain which then maps to a similar volume within the element. The two volumes would be shown to be the same.

In the following examples, we start with a unit cube element and make a simple modification and apply the Jacobian metric.

In all cases, we use 5 Gaussian points to sample the Jacobian measurements. This means that, for the hexahedron element, we sample the element at 125 points and report the most extreme value.

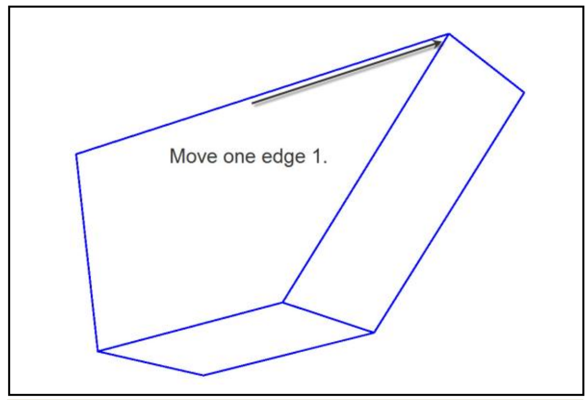


Fig. 5 Stretch: Jacobian Measure of 1.953

Starting with a cube element, select any or all of the nodes of a face and move them orthogonal to that face. All edges stretched by a factor of 2 have a Jacobian measure 2. The measure in the element above is not quite 2 because the Gaussian points used to sample the interior of the element are a small distance from the edge. If we measured along the edge we would see a measure of 2. Alternatively, if we stretch all four edges uniformly, we would see that the measure is exactly 2 throughout the interior.

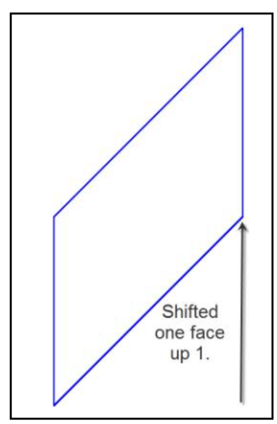


Fig. 6 Shear: Jacobian Measure

This example is a parallelepiped. The volume does not change under this mapping.

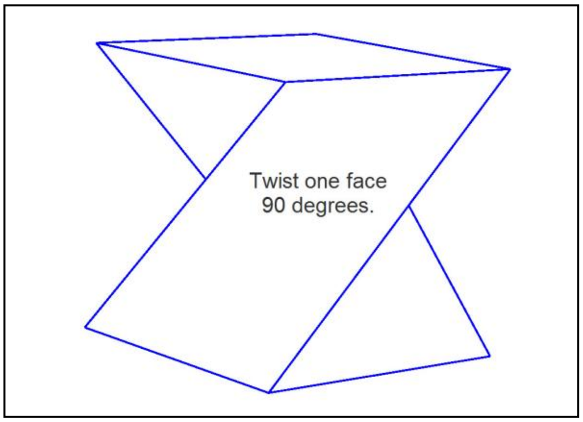


Fig. 7 Twist: Jacobian Measure of 1.414

It is remarkable how insensitive the measure is to a 90 degree twist. Consider how the volume of a region from the domain to the element changes when the element has a twist.

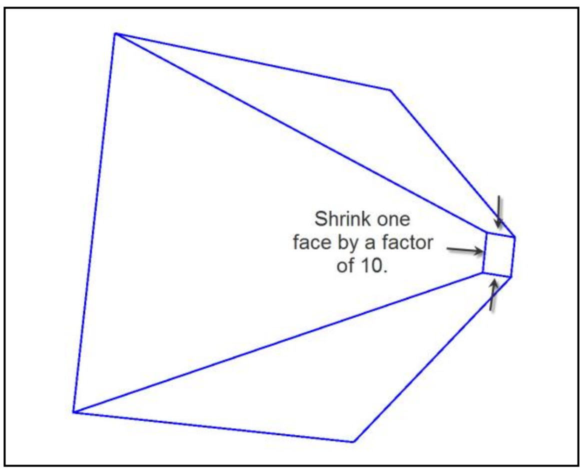


Fig. 8 Shrink: Jacobian Measure 7.031

This example shows how sensitive the Jacobian Metric is to aspect ratio. If we could measure closer to the boundary, we would see an even higher measure of distortion.

Related to this is the pyramid element. There is no unique way to handle pyramids. For example, it can be treated as a hexahedron with a degenerate face or as two tetrahedrons sharing a face. In the latter linear case, the shape of the base will probably match the face of an adjoining hexahedron, but only if the base is planar. Any movement in the simulation may cause the base to become non-planar, creating either gaps or overlaps between the two tetrahedrons and the adjoining hexahedron. Added to this problem is when quadratic (10 node tetrahedron and 20 node hexahedron) elements are employed. This will cause an additional node at the center of the base on the tetrahedron side that will not be matched by the adjoining quadratic hexahedron. If a tri-quadratic (27 node hexahedron) element adjoins the base of two quadratic (10 node) tetrahedrons, the discontinuity in the mesh will resemble the linear case. This is why the Jacobian measure in this paper treats the pyramid as a degenerate hexahedron.

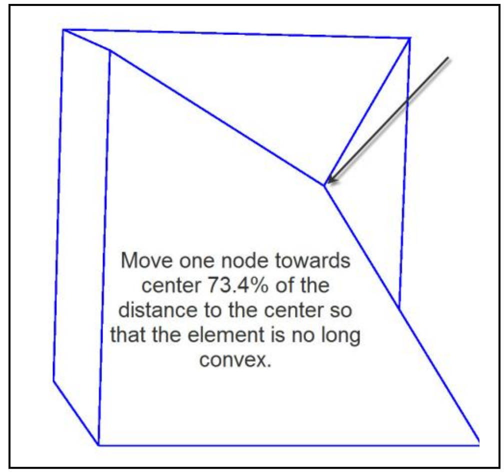


Fig. 9 Jacobian Measure of -.000126

In this last hexahedron example, we start out with a unit cube and make one simple modification. In this case, one node is moved along the line from the original position to the center of the cube. When the node is moved 73.4% of the way to the center, the element ceases to be convex and the Jacobian measure turns negative.

More generally, when the Jacobian measure is both positive and negative within an element, there will be a surface within the element where the Jacobian is zero. This is where the element volume folds onto itself. A better way to say this might be that if the Jacobian measure is both positive and negative within an element, there are at least two different points in the domain that map to the same point within the element.

We now turn our attention to some common distortions of a tetrahedron and how the Jacobian measure is affected.

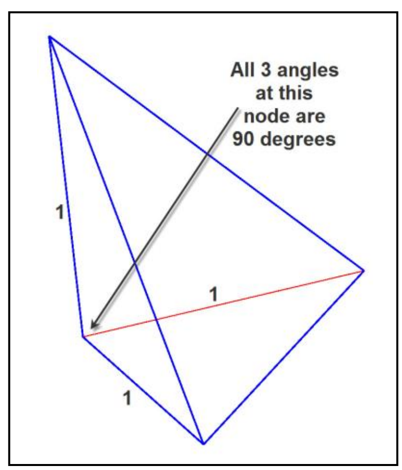


Fig 10 Ideal: Jacobian Measure 1.0

The above tetrahedron is ideal because it is similar to the shape of the domain. One of the nodes is met by edges that are 90 degrees to each other. These same edges must have the same lengths. In other words, the three triangles that meet at this node are identical right triangles.

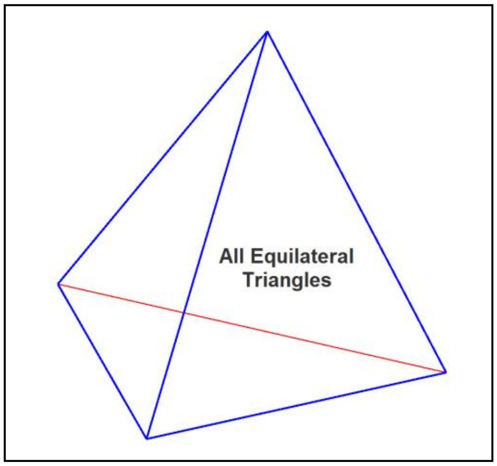


Fig. 11 Equal: Jacobian Measure of 2

What is remarkable about the tetrahedron in the above is that it has all 6 edges the same length (i.e. an aspect ratio of 1.0) and yet it has a relatively large Jacobian measure of 2.

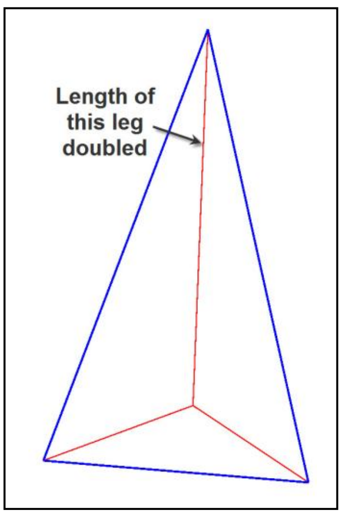


Fig. 12 Stretch: Jacobian Measure 1.05

What is even more remarkable is if we stretch one of the edges of the equilateral tetrahedron by doubling the edge length, the distortion in the element is almost imperceptible. This is because one of the nodes has some angles approaching 90 degrees, resembling the shape of the domain.

The next two tetrahedron examples are due to a simple distortion of the ideal tetrahedron, as described above.

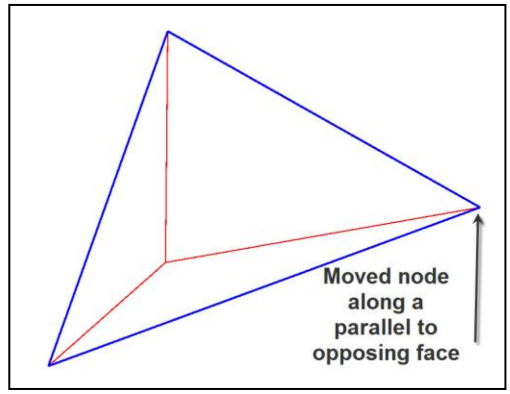


Fig. 13 Shear: Jacobian Measure .707

Figure 13 is an example of a measure less than 1. This is because the four nodes are getting closer to being co-planer.

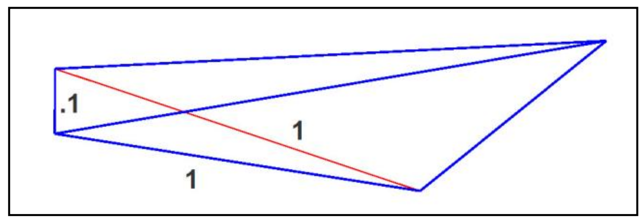


Fig. 14 Short Edge: Jacobian Measure of .2

Figure 14 is another example of the four nodes being nearly co-planar. The next two examples, found in some tetrahedral meshes, have extreme distortions and should be avoided.

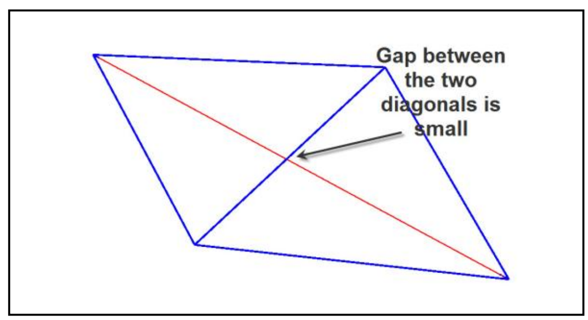


Fig. 15 Sliver: Jacobian Measure of .048

The element in Figure 15 is sometimes found in a mesh near the boundary. It is another example of the four nodes being nearly co-planer.

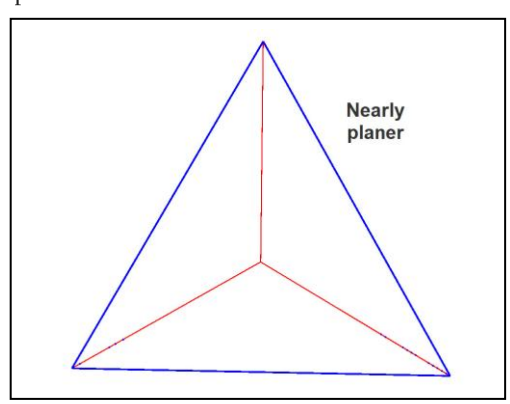


Fig. 16 Fuzzy: Jacobian Measure of ???

The example in Figure 15 was included because it is difficult to test. This is due to the limitations of a digital computer. The absolute value of the measure will be small and it is almost random whether the measure will be positive or negative.

4. Definition of the Standard Error Metric

We now define the Standard Error for the purpose of comparing the mesh quality of different versions of a finite element model where each version uses a different element type. This is based on the Jacobian Metric.

Thus far we have seen examples of the Jacobian metric for the interior of a single element. We now consider the Jacobian metric at each node of an element for all of the elements in a model.

For each node in the model, we get the Jacobian measure at that node for every element that contains the node. We define the Standard Error for that node to be the average of these Jacobian measurements. The Standard Error for the model will be the mean and standard deviation of the distribution of the average Jacobian measures for all the nodes in the model.

The Standard Error Metric produces a pair of numbers for every model, regardless the element type and mesh density.

5. Two Simple Examples

The first two examples are included due to their simplicity [9]. The hexahedron mesh is uniform except for a modest geometric progression in the mesh density along a cantilever beam. The tetrahedron mesh is derived from the hexahedron mesh by subdividing each hexahedron into 5 tetrahedron elements. At the lowest mesh density, the standard error measures an almost ideal and as the mesh density is increased, the standard error metric for the mesh hardly changes. This is the case for all of the element types.

Figure 17 shows the results for the resonance frequency calculation for a cantilever beam. Figure 18 shows the results for maximum stress of a cantilever beam. In this case, since the element quality is high in all cases, any inaccuracies in the calculations are not a function of the mesh quality.

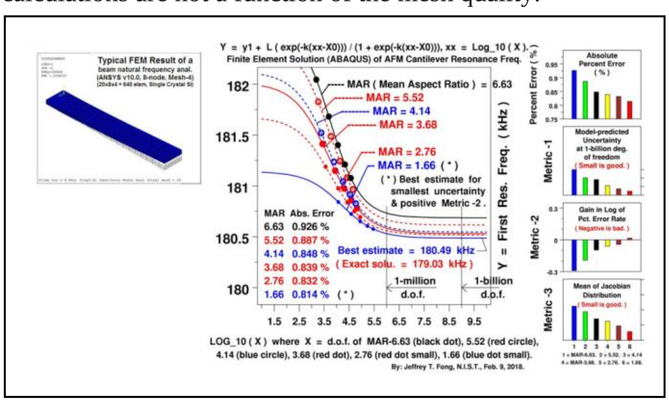


Fig. 17 Cantilever Beam Resonance

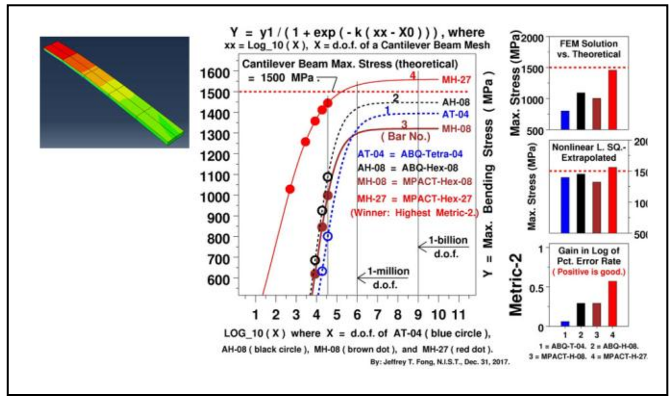
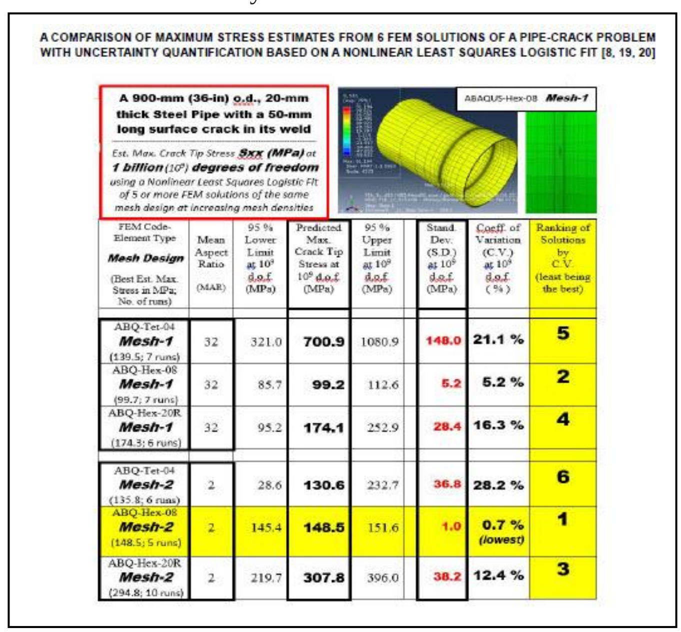


Fig. 18 Cantilever Beam Max Stess

6. An Example of a Pipe Elbow With a Crack

There was a surprise in this example, modeling a crack in a straight pipe using ABAQUS [11]. Mesh-1 (see Figure) has low mesh density, with three element types: 4 node tetrahedron, 8 node hexahedron, and 20 node hexahedron. Mesh-2 has high mesh density with the same 3 element types. One of the key features in both meshes is a transition in mesh density approaching the crack. The elements in the transition region usually have the lowest quality. Standard error indicated that the Mesh-1 with the 4 node tetrahedron mesh would fail. But the standard error did not distinguish between the 8 node and the 20 node hexahedrons using both Mesh-1 and Mesh-2. The mesh density, as in this case, may have no effect on the standard error. It is not clear why the 20 node hexahedron did so badly in both meshes.


A COMPARISON OF MAXIMUM STRESS ESTIMATES FROM 6 FEM SOLUTIONS OF A PIPE-CRACK PROBLEM WITH UNCERTAINTY QUANTIFICATION BASED ON A NONLINEAR LEAST SQUARES LOGISTIC FIT [8, 19, 20]

A 900-mm (36-in) o.d., 20-mm thick Steel Pipe with a 50-mm long surface crack in its weld

*Est. Max. Crack Tip Stress **Sxx (MPa)** at **1 billion** (10^9) **degrees of freedom** using a Nonlinear Least Squares Logistic Fit of 5 or more FEM solutions of the same mesh design at increasing mesh densities*

ABAQUS-Hex-08 ***Mesh-1***

FEM Code-Element Type **Mesh Design** (Best Est. Max. Stress in MPa; No. of runs)	Mean Aspect Ratio (MAR)	95 % Lower Limit at 10^9 d.o.f (MPa)	Predicted Max. Crack Tip Stress at 10^9 d.o.f (MPa)	95 % Upper Limit at 10^9 d.o.f (MPa)	Stand. Dev. (S.D.) at 10^9 d.o.f (MPa)	Coeff. of Variation (C.V.) at 10^9 d.o.f (%)	Ranking of Solutions by C.V. (least being the best)
ABQ-Tet-04 ***Mesh-1*** (139.5; 7 runs)	32	321.0	**700.9**	1080.9	**148.0**	**21.1 %**	**5**
ABQ-Hex-08 ***Mesh-1*** (99.7; 7 runs)	32	85.7	**99.2**	112.6	**5.2**	**5.2 %**	**2**
ABQ-Hex-20R ***Mesh-1*** (174.3; 6 runs)	32	95.2	**174.1**	252.9	**28.4**	**16.3 %**	**4**
ABQ-Tet-04 ***Mesh-2*** (135.8; 6 runs)	2	28.6	**130.6**	232.7	**36.8**	**28.2 %**	**6**
ABQ-Hex-08 ***Mesh-2*** (148.5; 5 runs)	2	145.4	**148.5**	151.6	**1.0**	**0.7 %** *(lowest)*	**1**
ABQ-Hex-20R ***Mesh-2*** (294.8; 10 runs)	2	219.7	**307.8**	396.0	**38.2**	**12.4 %**	**3**

Fig. 19 Cracked Pipe Calculations

7. An Example From Magnetic Resonance Imaging

In this next example, the MRI birdcage coil resonance is calculated using COSOL [6]. In Figure 20 two element arrangements were used for comparison. Mesh-1 consists of quadratic tetrahedrons. Mesh-2 consists of a mixture of tri-quadratic hexahedron, quadratic pyramid, and quadratic tetrahedron elements. Mesh-1 proved to be more accurate. This was done by analyzing 7 versions of each mesh arrangement by increasing the mesh density.

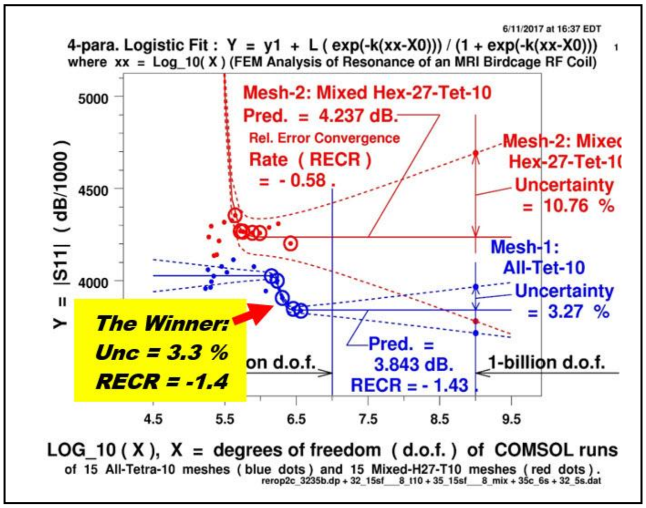


Fig. 20 Comparison of COMSOL meshes

Table 1 shows the numeric results of these calculations for Mesh-1. Table 2 shows the standard error metric (both mean and standard deviation) for these 7 meshes.

	Mesh No.	Refine Para-meter	Degree of Freedom	Resonant Frequency (MHz)	S11 (dB)
M102	2	0.90	183,408	19.652	-4.060869
M103	3	0.80	199,594	19.652	-3.964955
M104	4	0.70	205,312	19.644	-3.996195
M105	5	0.60	221,120	19.626	-4.026074
M106	6	0.50	284,826	19.597	-4.078074
M107	7	0.40	337,660	19.589	-4.046671
M108	8	0.35	415,914	19.558	-4.114915

Table 1 COMSOL Mesh-1 Results

	M102	M103	M104	M105	M106	M107	M107
Mean	.9940	1.000	.9841	1.001	1.034	1.089	1.089
S.D.	.4427	.4597	.4625	.477	.5005	.5669	.6015

Table 2 Standard Error Metrics for Mesh-1

8. Conclusions

An unexpected result from this study is that a tetrahedral element, with aspect ratio of 1, is far from the ideal tetrahedron. In fact, the best tetrahedral element comes from the corner region of a quadratic hexahedral element. The MRI example using quadratic tetrahedron demonstrates high quality elements with the mean hovering around 1. The fact that this series of calculations resulted in the highest level of accuracy supports the use of the Standard Error to aid in the *a priori* test for accuracy.

The Standard Error Metric is based entirely on geometry and is independent of the analysis platform. Except for some simple examples, it is not possible that the Standard Error Metric can be used solely to predict the accuracy of a simulation. It does not incorporate the need for a higher quality mesh locally due to the effect of boundary conditions, constraints, and loads. It also does not include the behavior of the materials and any possible non-linearity.

As was pointed out, the Standard Error is insensitive to mesh density. It was also pointed out that the quadratic form of an element type produces the identical Standard Error as the linear version of the element (only in the initial state when the mid-nodes are located at the mid-points of the edges of the quadratic element). In most cases, as the mesh density is increased or when the element type is changed from linear to quadratic, the solution offered by the simulation is more accurate.

This is compelling evidence that no *a piori* metric of the mesh can predict the accuracy of a simulation without taking into consideration all of these properties of a simulation.

Methods to show accuracy of a simulation that are of the *a posteriori* type have more information about the simulation. Multiple runs with increasing mesh density have a good chance of showing convergence. But such methods have a serious weakness. When a sequence of improved meshes fails to converge, the *a posteriori* methods cannot offer a remedy.

Both the *a posteriori* methods and the Standard Error are global in nature. This is not sufficient. For example, the Standard Error is insensitive to a sheared mesh or mesh density. It also is unaffected by irregular nodes [7] or rapid change in mesh density, both which can affect the simulation of shocks and fluid flow, in particular. It is possible that a small region of the mesh is of poor quality, but it has little effect on the Standard Error and a huge negative effect on the simulation. Only local mesh quality tests can detect these problems in the mesh.

The Standard Error Metric would be more useful in predicting the accuracy of a simulation if the average measure was weighted locally and automatically, based on the importance of each region of the mesh to the accuracy of the simulation.

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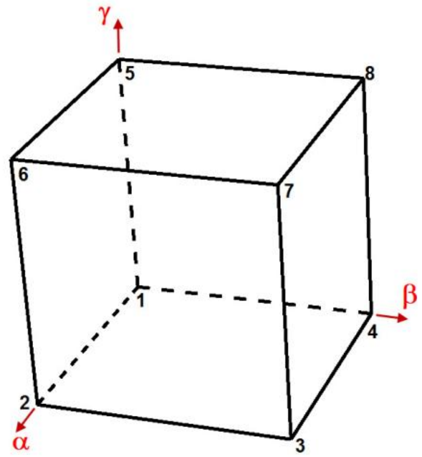
9. Appendix – Finite Element Node Numbering, Shape Functions, and Their Derivatives

Linear 8 Node Hexahedron



$$-1 \le \alpha,\beta,\gamma \le 1$$

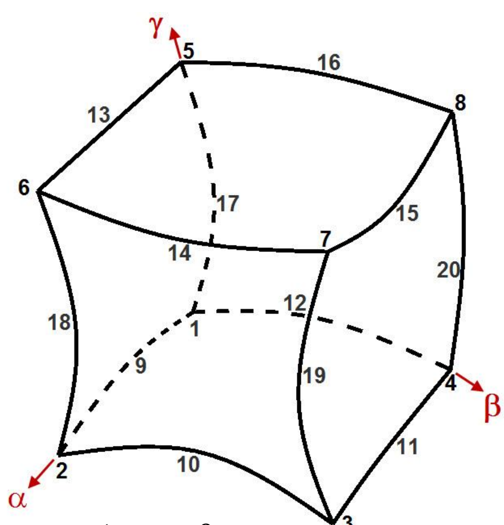
$$N_1 = (1-\alpha)(1-\beta)(1-\gamma)/8$$
$$N_2 = (1+\alpha)(1-\beta)(1-\gamma)/8$$
$$N_3 = (1+\alpha)(1+\beta)(1-\gamma)/8$$
$$N_4 = (1-\alpha)(1+\beta)(1-\gamma)/8$$
$$N_5 = (1-\alpha)(1-\beta)(1+\gamma)/8$$
$$N_6 = (1+\alpha)(1-\beta)(1+\gamma)/8$$
$$N_7 = (1+\alpha)(1+\beta)(1+\gamma)/8$$
$$N_8 = (1-\alpha)(1+\beta)(1+\gamma)/8$$

$$\frac{\partial N_1}{\partial \alpha} = -(1-\beta)(1-\gamma)/8$$
$$\frac{\partial N_2}{\partial \alpha} = (1-\beta)(1-\gamma)/8$$
$$\frac{\partial N_3}{\partial \alpha} = (1+\beta)(1-\gamma)/8$$
$$\frac{\partial N_4}{\partial \alpha} = -(1+\beta)(1-\gamma)/8$$
$$\frac{\partial N_5}{\partial \alpha} = -(1-\beta)(1+\gamma)/8$$
$$\frac{\partial N_6}{\partial \alpha} = (1-\beta)(1+\gamma)/8$$
$$\frac{\partial N_7}{\partial \alpha} = (1+\beta)(1+\gamma)/8$$
$$\frac{\partial N_8}{\partial \alpha} = -(1+\beta)(1+\gamma)/8$$

$$\frac{\partial N_1}{\partial \beta} = -(1-\alpha)(1-\gamma)/8$$
$$\frac{\partial N_2}{\partial \beta} = -(1+\alpha)(1-\gamma)/8$$
$$\frac{\partial N_3}{\partial \beta} = (1-\alpha)(1-\gamma)/8$$
$$\frac{\partial N_4}{\partial \beta} = (1+\alpha)(1-\gamma)/8$$
$$\frac{\partial N_5}{\partial \beta} = -(1-\alpha)(1+\gamma)/8$$
$$\frac{\partial N_6}{\partial \beta} = -(1+\alpha)(1+\gamma)/8$$
$$\frac{\partial N_7}{\partial \beta} = (1-\alpha)(1+\gamma)/8$$
$$\frac{\partial N_8}{\partial \beta} = (1+\alpha)(1+\gamma)/8$$

$$\frac{\partial N_1}{\partial \gamma} = -(1-\alpha)(1-\beta)/8$$
$$\frac{\partial N_2}{\partial \gamma} = -(1+\alpha)(1-\beta)/8$$
$$\frac{\partial N_3}{\partial \gamma} = -(1-\alpha)(1+\beta)/8$$
$$\frac{\partial N_4}{\partial \gamma} = -(1+\alpha)(1+\beta)/8$$
$$\frac{\partial N_5}{\partial \gamma} = (1-\alpha)(1-\beta)/8$$
$$\frac{\partial N_6}{\partial \gamma} = (1+\alpha)(1-\beta)/8$$
$$\frac{\partial N_7}{\partial \gamma} = (1-\alpha)(1+\beta)/8$$
$$\frac{\partial N_8}{\partial \gamma} = (1+\alpha)(1+\beta)/8$$

Quadratic 20 Node Hexahedron



$$-1 \le \alpha,\beta,\gamma$$

$$N_1 = (1-\alpha)\ast(1-\beta)\ast(1-\gamma)\ast(-2-\alpha-\beta-\gamma)/8$$
$$N_2 = (1+\alpha)\ast(1-\beta)\ast(1-\gamma)\ast(-2+\alpha-\beta-\gamma)/8$$
$$N_3 = (1+\alpha)\ast(1+\beta)\ast(1-\gamma)\ast(-2+\alpha+\beta-\gamma)/8$$
$$N_4 = (1-\alpha)\ast(1+\beta)\ast(1-\gamma)\ast(-2-\alpha+\beta-\gamma)/8$$
$$N_5 = (1-\alpha)\ast(1-\beta)\ast(1+\gamma)\ast(-2-\alpha-\beta+\gamma)/8$$
$$N_6 = (1+\alpha)\ast(1-\beta)\ast(1+\gamma)\ast(-2+\alpha-\beta+\gamma)/8$$
$$N_7 = (1+\alpha)\ast(1+\beta)\ast(1+\gamma)\ast(-2+\alpha+\beta+\gamma)/8$$
$$N_8 = (1-\alpha)\ast(1+\beta)\ast(1+\gamma)\ast(-2-\alpha+\beta+\gamma)/8$$
$$N_9 = (1-\alpha\ast\alpha)\ast(1-\beta)\ast(1-\gamma)/4$$
$$N_{10} = (1+\alpha)\ast(1-\beta\ast\beta)\ast(1-\gamma)$$

$$\frac{\partial N_1}{\partial \alpha} = -((1-\beta)\ast(1-\gamma)\ast(-\alpha-\beta-\gamma-2)+(1-\alpha)\ast(1-\beta)\ast(1-\gamma))/8$$
$$\frac{\partial N_2}{\partial \alpha} = ((1-\beta)\ast(1-\gamma)\ast(\alpha-\beta-\gamma-2)+(1+\alpha)\ast(1-\beta)\ast(1-\gamma))/8$$
$$\frac{\partial N_3}{\partial \alpha} = ((1+\beta)\ast(1-\gamma)\ast(\alpha+\beta-\gamma-2)+(1+\alpha)\ast(1+\beta)\ast(1-\gamma))/8$$
$$\frac{\partial N_4}{\partial \alpha} = -((1+\beta)\ast(1-\gamma)\ast(-\alpha+\beta-\gamma-2)+(1-\alpha)\ast(1+\beta)\ast(1-\gamma))/8$$
$$\frac{\partial N_5}{\partial \alpha} = -((1-\beta)\ast(1+\gamma)\ast(-\alpha-\beta+\gamma-2)+(1-\alpha)\ast(1-\beta)\ast(1+\gamma))/8$$
$$\frac{\partial N_6}{\partial \alpha} = ((1-\beta)\ast(1+\gamma)\ast(\alpha-\beta+\gamma-2)+(1+\alpha)\ast(1-\beta)\ast(1+\gamma))/8$$
$$\frac{\partial N_7}{\partial \alpha} = ((1+\beta)\ast(1+\gamma)\ast(\alpha+\beta+\gamma-2)+(1+\alpha)\ast(1+\beta)\ast(1+\gamma))/8$$
$$\frac{\partial N_8}{\partial \alpha} = -((1+\beta)\ast(1+\gamma)\ast(-\alpha+\beta+\gamma-2)+(1-\alpha)\ast(1+\beta)\ast(1+\gamma))/8$$
$$\frac{\partial N_9}{\partial \alpha} = -\alpha\ast(1-\beta)\ast(1-\gamma)/2$$
$$\frac{\partial N_{10}}{\partial \alpha} = (1-\beta\ast\beta)\ast(1-\gamma)/4$$
$$\frac{\partial N_{11}}{\partial \alpha} = -\alpha\ast(1+\beta)\ast(1-\gamma)/2$$
$$\frac{\partial N_{12}}{\partial \alpha} = -(1-\beta\ast\beta)\ast(1-\gamma)/4$$
$$\frac{\partial N_{13}}{\partial \alpha} = -\alpha\ast(1-\beta)\ast(1+\gamma)/2$$
$$\frac{\partial N_{14}}{\partial \alpha} = (1-\beta\ast\beta)\ast(1+\gamma)/4$$
$$\frac{\partial N_{15}}{\partial \alpha} = -\alpha\ast(1+\beta)\ast(1+\gamma)/2$$

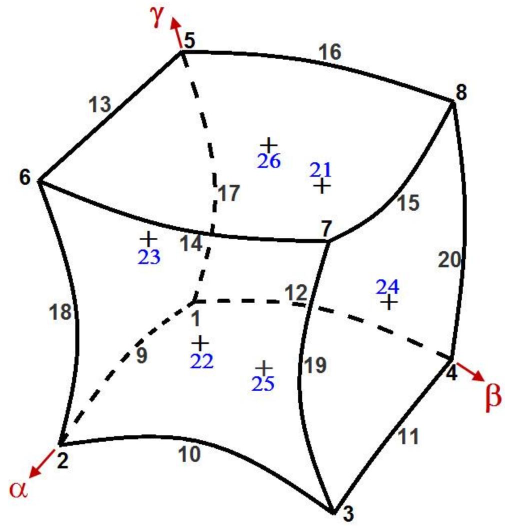
$$N_{11} = (1-\alpha*\alpha)*(1+\beta)*(1-\gamma)/4$$
$$N_{12} = (1-\alpha)*(1-\beta*\beta)*(1-\gamma)/4$$
$$N_{13} = (1-\alpha)*(1-\beta)*(1-\gamma*\gamma)/4$$
$$N_{14} = (1+\alpha)*(1-\beta)*(1-\gamma*\gamma)/4$$
$$N_{15} = (1+\alpha)*(1+\beta)*(1-\gamma*\gamma)/4$$
$$N_{16} = (1-\alpha)*(1+\beta)*(1-\gamma*\gamma)/4$$
$$N_{17} = (1-\alpha*\alpha)*(1-\beta)*(1+\gamma)/4$$
$$N_{18} = (1+\alpha)*(1-\beta*\beta)*(1+\gamma)/4$$
$$N_{19} = (1-\alpha*\alpha)*(1+\beta)*(1+\gamma)/4$$
$$N_{20} = (1-\alpha)*(1-\beta*\beta)*(1+\gamma)/4$$

$$\frac{\partial N_{16}}{\partial\alpha} = -(1-\beta*\beta)*(1+\gamma)/4$$
$$\frac{\partial N_{17}}{\partial\alpha} = -(1-\gamma*\gamma)*(1-\beta)/4$$
$$\frac{\partial N_{18}}{\partial x} = (1-\gamma*\gamma)*(1-\beta)/4$$
$$\frac{\partial N_{19}}{\partial\alpha} = (1-\gamma*\gamma)*(1+\beta)/4$$
$$\frac{\partial N_{20}}{\partial\alpha} = -(1-\gamma*\gamma)*(1+\beta)/4$$

$$\frac{\partial N_1}{\partial\beta} = -((1-\alpha)*(1-\gamma)*(-\alpha-\beta-\gamma-2)+(1-\alpha)*(1-\beta)*(1-\gamma))/8$$
$$\frac{\partial N_2}{\partial\beta} = -((1+\alpha)*(1-\gamma)*(\alpha-\beta-\gamma-2)+(1+\alpha)*(1-\beta)*(1-\gamma))/8$$
$$\frac{\partial N_3}{\partial\beta} = ((1+\alpha)*(1-\gamma)*(\alpha+\beta-\gamma-2)+(1+\alpha)*(1+\beta)*(1-\gamma))/8$$
$$\frac{\partial N_4}{\partial\beta} = ((1-\alpha)*(1-\gamma)*(-\alpha+\beta-\gamma-2)+(1-\alpha)*(1+\beta)*(1-\gamma))/8$$
$$\frac{\partial N_5}{\partial\beta} = -((1-\alpha)*(1+\gamma)*(-\alpha-\beta+\gamma-2)+(1-\alpha)*(1-\beta)*(1+\gamma))/8$$
$$\frac{\partial N_6}{\partial\beta} = -((1+\alpha)*(1+\gamma)*(\alpha-\beta+\gamma-2)+(1+\alpha)*(1-\beta)*(1+\gamma))/8$$
$$\frac{\partial N_7}{\partial\beta} = ((1+\alpha)*(1+\gamma)*(\alpha+\beta+\gamma-2)+(1+\alpha)*(1+\beta)*(1+\gamma))/8$$
$$\frac{\partial N_8}{\partial\beta} = ((1-\alpha)*(1+\gamma)*(-\alpha+\beta+\gamma-2)+(1-\alpha)*(1+\beta)*(1+\gamma))/8$$
$$\frac{\partial N_9}{\partial\beta} = -(1-\alpha*\alpha)*(1-\gamma)/4$$
$$\frac{\partial N_{10}}{\partial\beta} = -\beta*(1+\alpha)*(1-\gamma)/2$$
$$\frac{\partial N_{11}}{\partial\beta} = (1-\alpha*\alpha)*(1-\gamma)/4$$
$$\frac{\partial N_{12}}{\partial\beta} = -\beta*(1-\alpha)*(1-\gamma)/2$$
$$\frac{\partial N_{13}}{\partial\beta} = -(1-\alpha*\alpha)*(1+\gamma)/4$$
$$\frac{\partial N_{14}}{\partial\beta} = -\beta*(1+\alpha)*(1+\gamma)/2$$
$$\frac{\partial N_{15}}{\partial\beta} = (1-\alpha*\alpha)*(1+\gamma)/4$$
$$\frac{\partial N_{16}}{\partial\beta} = -\beta*(1-\alpha)*(1+\gamma)/2$$
$$\frac{\partial N_{17}}{\partial\beta} = -(1-\gamma*\gamma)*(1-\alpha)/4$$
$$\frac{\partial N_{18}}{\partial\beta} = -(1-\gamma*\gamma)*(1+\alpha)/4$$
$$\frac{\partial N_{19}}{\partial\beta} = (1-\gamma*\gamma)*(1+\alpha)/4$$
$$\frac{\partial N_{20}}{\partial\beta} = (1-\gamma*\gamma)*(1-\alpha)/4$$

$$\frac{\partial N_1}{\partial\gamma} = -((1-\alpha)*(1-\beta)*(-\alpha-\beta-\gamma-2)+(1-\alpha)*(1-\beta)*(1-\gamma))/8$$
$$\frac{\partial N_2}{\partial\gamma} = -((1+\alpha)*(1-\beta)*(\alpha-\beta-\gamma-2)+(1+\alpha)*(1-\beta)*(1-\gamma))/8$$
$$\frac{\partial N_3}{\partial\gamma} = -((1+\alpha)*(1+\beta)*(\alpha+\beta-\gamma-2)+(1+\alpha)*(1+\beta)*(1-\gamma))/8$$
$$\frac{\partial N_4}{\partial\gamma} = -((1-\alpha)*(1+\beta)*(-\alpha+\beta-\gamma-2)+(1-\alpha)*(1+\beta)*(1-\gamma))/8$$
$$\frac{\partial N_5}{\partial\gamma} = ((1-\alpha)*(1-\beta)*(-\alpha-\beta+\gamma-2)+(1-\alpha)*(1-\beta)*(1+\gamma))/8$$
$$\frac{\partial N_6}{\partial\gamma} = ((1+\alpha)*(1-\beta)*(\alpha-\beta+\gamma-2)+(1+\alpha)*(1-\beta)*(1+\gamma))/8$$
$$\frac{\partial N_7}{\partial\gamma} = ((1+\alpha)*(1+\beta)*(\alpha+\beta+\gamma-2)+(1+\alpha)*(1+\beta)*(1+\gamma))/8$$
$$\frac{\partial N_8}{\partial\gamma} = ((1-\alpha)*(1+\beta)*(-\alpha+\beta+\gamma-2)+(1-\alpha)*(1+\beta)*(1+\gamma))/8$$
$$\frac{\partial N_9}{\partial\gamma} = -(1-\alpha*\alpha)*(1-\beta)/4$$
$$\frac{\partial N_{10}}{\partial\gamma} = -(1-\beta*\beta)*(1+\alpha)/4$$
$$\frac{\partial N_{11}}{\partial\gamma} = -(1-\alpha*\alpha)*(1+\beta)/4$$
$$\frac{\partial N_{12}}{\partial\gamma} = -(1-\beta*\beta)*(1-\alpha)/4$$
$$\frac{\partial N_{13}}{\partial\gamma} = (1-\alpha*\alpha)*(1-\beta)/4$$
$$\frac{\partial N_{14}}{\partial\gamma} = (1-\beta*\beta)*(1+\alpha)/4$$
$$\frac{\partial N_{15}}{\partial\gamma} = (1-\alpha*\alpha)*(1+\beta)/4$$
$$\frac{\partial N_{16}}{\partial\gamma} = (1-\beta*\beta)*(1-\alpha)/4$$
$$\frac{\partial N_{17}}{\partial\gamma} = -\gamma*(1-\alpha)*(1-\beta)/2$$
$$\frac{\partial N_{18}}{\partial\gamma} = -\gamma*(1+\alpha)*(1-\beta)/2$$
$$\frac{\partial N_{19}}{\partial\gamma} = -\gamma*(1+\alpha)*(1+\beta)/2$$
$$\frac{\partial N_{20}}{\partial\gamma} = -\gamma*(1-\alpha)*(1+\beta)/2$$

Tri-quadratic 27 Node Hexahedron



The blue node numbers are mid-face nodes indicated by a cross. Node 27 (not shown) is located at the center of the element.

$$-1 \le \alpha, \beta, \gamma \le 1$$

$N_1 = \alpha \ast (\alpha - 1) \ast \beta \ast (\beta - 1) \ast \gamma \ast (\gamma - 1)/8$

$N_2 = \alpha \ast (\alpha + 1) \ast \beta \ast (\beta - \beta \ast \gamma \ast (\gamma - 1)/8$

$N_3 = \alpha \ast (\alpha + 1) \ast \beta \ast (\beta + 1) \ast \gamma \ast (\gamma - 1)/8$

$N_4 = \alpha \ast (\alpha - 1) \ast \beta \ast (\beta + 1) \ast \gamma \ast (\gamma - 1)/8$

$N_5 = \alpha \ast (\alpha - 1) \ast \beta \ast (\beta - 1) \ast \gamma \ast (\gamma + 1)/8$

$N_6 = \alpha \ast (\alpha + 1) \ast \beta \ast (\beta - 1) \ast \gamma \ast (\gamma + 1)/8$

$N_7 = \alpha \ast (\alpha + 1) \ast \beta \ast (\beta + 1) \ast \gamma \ast (\gamma + 1)/8$

$N_8 = \alpha \ast (\alpha - 1) \ast \beta \ast (\beta + 1) \ast \gamma \ast (\gamma + 1)/8$

$N_9 = (1 - \alpha \ast \alpha) \ast \beta \ast (\beta - 1) \ast \gamma \ast (\gamma - 1)/4$

$N_{10} = \alpha \ast (\alpha - 1) \ast (1 - \beta \ast \beta) \ast \gamma \ast (\gamma - 1)/4$

$N_{11} = (1 - \alpha \ast \alpha) \ast \beta \ast (\beta + 1) \ast \gamma \ast (\gamma - 1)/4$

$N_{12} = \alpha \ast (\alpha - 1) \ast (1 - \beta \ast \beta) \ast \gamma \ast (\gamma - 1)/4$

$N_{13} = (1 - \alpha \ast \alpha) \ast \beta \ast (\beta - 1) \ast \gamma \ast (\gamma + 1)/4$

$N_{14} = \alpha \ast (\alpha + 1) \ast (1 - \beta \ast \beta) \ast \gamma \ast (\gamma + 1)/4$

$N_{15} = (1 - \alpha \ast \alpha) \ast \beta \ast (\beta + 1) \ast \gamma \ast (\gamma + 1)/4$

$N_{16} = \alpha \ast (\alpha - 1) \ast (1 - \beta \ast \beta) \ast \gamma \ast (\gamma + 1)/4$

$N_{17} = \alpha \ast (\alpha - 1) \ast \beta \ast (\beta - 1) \ast (1 - \gamma \ast \gamma)/4$

$N_{18} = \alpha \ast (\alpha + 1) \ast \beta \ast (\beta - 1) \ast (1 - \gamma \ast \gamma)/4$

$N_{19} = \alpha \ast (\alpha + 1) \ast \beta \ast (\beta + 1) \ast (1 - \gamma \ast \gamma)/4$

$N_{20} = \alpha \ast (\alpha - 1) \ast \beta \ast (\beta + 1) \ast (1 - \gamma \ast \gamma)/4$

$N_{21} = \alpha \ast (\alpha - 1) \ast (1 - \beta \ast \beta) \ast (1 - \gamma \ast \gamma)/2$

$N_{22} = \alpha \ast (\alpha + 1) \ast (1 - \beta \ast \beta) \ast (1 - \gamma \ast \gamma)/2$

$N_{23} = (1 - \alpha \ast \alpha) \ast \beta \ast (\beta - 1) \ast (1 - \gamma \ast \gamma)/2$

$N_{24} = (1 - \alpha \ast \alpha) \ast \beta \ast (\beta + 1) \ast (1 - \gamma \ast \gamma)/2$

$N_{25} = (1 - \alpha \ast \alpha) \ast (1 - \beta \ast \beta) \ast \gamma \ast (\gamma - 1)/2$

$N_{26} = (1 - \alpha \ast \alpha) \ast (1 - \beta \ast \beta) \ast \gamma \ast (\gamma + 1)/2$

$N_{27} = (1 - \alpha \ast \alpha) \ast (1 - \beta \ast \beta) \ast (1 - \gamma \ast \gamma)$

$$\frac{\partial N_1}{\partial \alpha} = (2 \ast \alpha - 1) \ast \beta \ast (\beta - 1) \ast \gamma \ast (\gamma - 1)/8$$

$$\frac{\partial N_2}{\partial \alpha} = (2 \ast \alpha + 1) \ast \beta \ast (\beta - 1) \ast \gamma \ast (\gamma - 1)/8$$

$$\frac{\partial N_3}{\partial \alpha} = (2 \ast \alpha + 1) \ast \beta \ast (\beta + 1) \ast \gamma \ast (\gamma - 1)/8$$

$$\frac{\partial N_4}{\partial \alpha} = (2 \ast \alpha - 1) \ast \beta \ast (\beta + 1) \ast \gamma \ast (\gamma - 1)/8$$

$$\frac{\partial N_5}{\partial \alpha} = (2 \ast \alpha - 1) \ast \beta \ast (\beta - 1) \ast \gamma \ast (\gamma + 1)/8$$

$$\frac{\partial N_6}{\partial \alpha} = (2 \ast \alpha + 1) \ast \beta \ast (\beta - 1) \ast \gamma \ast (\gamma + 1)/8$$

$$\frac{\partial N_7}{\partial \alpha} = (2 \ast \alpha + 1) \ast \beta \ast (\beta + 1) \ast \gamma \ast (\gamma + 1)/8$$

$$\frac{\partial N_8}{\partial \alpha} = (2 \ast \alpha - 1) \ast \beta \ast (\beta + 1) \ast \gamma \ast (\gamma + 1)/8$$

$$\frac{\partial N_9}{\partial \alpha} = -\alpha \ast \beta \ast (\beta - 1) \ast \gamma \ast (\gamma - 1)/2$$

$$\frac{\partial N_{10}}{\partial \alpha} = (2 \ast \alpha + 1) \ast (1 - \beta \ast \beta) \ast \gamma \ast (\gamma - 1)/4$$

$$\frac{\partial N_{11}}{\partial \alpha} = -\alpha \ast \beta \ast (\beta + 1) \ast \gamma \ast (\gamma - 1)/2$$

$$\frac{\partial N_{12}}{\partial \alpha} = (2 \ast \alpha - 1) \ast (1 - \beta \ast \beta) \ast \gamma \ast (\gamma - 1)/4$$

$$\frac{\partial N_{13}}{\partial \alpha} = -\alpha \ast \beta \ast (\beta - 1) \ast \gamma \ast (\gamma + 1)/2$$

$$\frac{\partial N_{14}}{\partial \alpha} = (2 \ast \alpha + 1) \ast (1 - \beta \ast \beta) \ast \gamma \ast (\gamma + 1)/4$$

$$\frac{\partial N_{15}}{\partial \alpha} = -\alpha \ast \beta \ast (\beta + 1) \ast \gamma \ast (\gamma + 1)/2$$

$$\frac{\partial N_{16}}{\partial \alpha} = (2 \ast \alpha - 1) \ast (1 - \beta \ast \beta) \ast \gamma \ast (\gamma + 1)/4$$

$$\frac{\partial N_{17}}{\partial \alpha} = (2 \ast \alpha - 1) \ast \beta \ast (\beta - 1) \ast (1 - \gamma \ast \gamma)/4$$

$$\frac{\partial N_{18}}{\partial \alpha} = (2 \ast \alpha + 1) \ast \beta \ast (\beta - 1) \ast (1 - \gamma \ast \gamma)/4$$

$$\frac{\partial N_{19}}{\partial \alpha} = (2 \ast \alpha + 1) \ast \beta \ast (\beta + 1) \ast (1 - \gamma \ast \gamma)/4$$

$$\frac{\partial N_{20}}{\partial \alpha} = (2 \ast \alpha - 1) \ast \beta \ast (\beta + 1) \ast (1 - \gamma \ast \gamma)/$$

$$\frac{\partial N_{21}}{\partial \alpha} = (2 \ast \alpha - 1) \ast (1 - \beta \ast \beta) \ast (1 - \gamma \ast \gamma)/2$$

$$\frac{\partial N_{22}}{\partial \alpha} = (2 \ast \alpha + 1) \ast (1 - \beta \ast \beta) \ast (1 - \gamma \ast \gamma)/2$$

$$\frac{\partial N_{23}}{\partial \alpha} = -\alpha \ast \beta \ast (\beta - 1) \ast (1 - \gamma \ast \gamma)$$

$$\frac{\partial N_{24}}{\partial \alpha} = -\alpha \ast \beta \ast (\beta + 1) \ast (1 - \gamma \ast \gamma)$$

$$\frac{\partial N_{25}}{\partial \alpha} = -\alpha \ast (1 - \beta \ast \beta) \ast \gamma \ast (\gamma - 1)$$

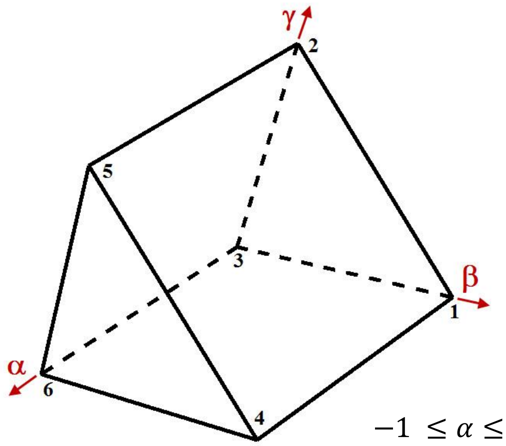
$$\frac{\partial N_{26}}{\partial \alpha} = -\alpha \ast (1 - \beta \ast \beta) \ast \gamma \ast (\gamma + 1)$$

$$\frac{\partial N_{27}}{\partial \alpha} = -2 \ast \alpha \ast (1 - \beta \ast \beta) \ast (1 - \gamma \ast \gamma)$$

$$\frac{\partial N_1}{\partial \beta} = \alpha * (\alpha - 1) * (2 * \beta - 1) * \gamma * (\gamma - 1)/8$$
$$\frac{\partial N_2}{\partial \beta} = \alpha * (\alpha + 1) * (2 * \beta - 1) * \gamma * (\gamma - 1)/8$$
$$\frac{\partial N_3}{\partial \beta} = \alpha * (\alpha + 1) * (2 * \beta + 1) * \gamma * (\gamma - 1)/8$$
$$\frac{\partial N_4}{\partial \beta} = \alpha * (\alpha - 1) * (2 * \beta + 1) * \gamma * (\gamma - 1)/8$$
$$\frac{\partial N_5}{\partial \beta} = \alpha * (\alpha - 1) * (2 * \beta - 1) * \gamma * (\gamma + 1)/8$$
$$\frac{\partial N_6}{\partial \beta} = \alpha * (\alpha + 1) * (2 * \beta - 1) * \gamma * (\gamma + 1)/8$$
$$\frac{\partial N_7}{\partial \beta} = \alpha * (\alpha + 1) * (2 * \beta + 1) * \gamma * (\gamma + 1)/8$$
$$\frac{\partial N_8}{\partial \beta} = \alpha * (\alpha - 1) * (2 * \beta + 1) * \gamma * (\gamma + 1)/8$$
$$\frac{\partial N_9}{\partial \beta} = (1 - \alpha * \alpha) * (2 * \beta - 1) * \gamma * (\gamma - 1)/4$$
$$\frac{\partial N_{10}}{\partial \beta} = -\alpha * (\alpha + 1) * \beta * \gamma * (\gamma - 1)/2$$
$$\frac{\partial N_{11}}{\partial \beta} = (1 - \alpha * \alpha) * (2 * \beta + 1) * \gamma * (\gamma - 1)/4$$
$$\frac{\partial N_{12}}{\partial \beta} = -\alpha * (\alpha - 1) * \beta * \gamma * (\gamma - 1)/2$$
$$\frac{\partial N_{13}}{\partial \beta} = (1 - \alpha * \alpha) * (2 * \beta - 1) * \gamma * (\gamma + 1)/4$$
$$\frac{\partial N_{14}}{\partial \beta} = -\alpha * (\alpha + 1) * \beta * \gamma * (\gamma + 1)/2$$
$$\frac{\partial N_{15}}{\partial \beta} = (1 - \alpha * \alpha) * (2 * \beta + 1) * \gamma * (\gamma + 1)/4$$
$$\frac{\partial N_{16}}{\partial \beta} = -\alpha * (\alpha - 1) * \beta * \gamma * (\gamma + 1)/2$$
$$\frac{\partial N_{17}}{\partial \beta} = \alpha * (\alpha - 1) * (2 * \beta - 1) * (1 - \gamma * \gamma)/4$$
$$\frac{\partial N_{18}}{\partial \beta} = \alpha * (\alpha + 1) * (2 * \beta - 1) * (1 - \gamma * \gamma)/4$$
$$\frac{\partial N_{19}}{\partial \beta} = \alpha * (\alpha + 1) * (2 * \beta + 1) * (1 - \gamma * \gamma)/4$$
$$\frac{\partial N_{20}}{\partial \beta} = \alpha * (\alpha - 1) * (2 * \beta + 1) * (1 - \gamma * \gamma)/4$$
$$\frac{\partial N_{21}}{\partial \beta} = -\alpha * (\alpha - 1) * \beta * (1 - \gamma * \gamma)$$
$$\frac{\partial N_{22}}{\partial \beta} = -\alpha * (\alpha + 1) * \beta * (1 - \gamma * \gamma)$$
$$\frac{\partial N_{23}}{\partial \beta} = (1 - \alpha * \alpha) * (2 * \beta - 1) * (1 - \gamma * \gamma)/2$$
$$\frac{\partial N_{24}}{\partial \beta} = (1 - \alpha * \alpha) * (2 * \beta + 1) * (1 - \gamma * \gamma)/2$$
$$\frac{\partial N_{25}}{\partial \beta} = -(1 - \alpha * \alpha) * \beta * \gamma * (\gamma - 1)$$
$$\frac{\partial N_{26}}{\partial \beta} = -(1 - \alpha * \alpha) * \beta * \gamma * (\gamma + 1)$$
$$\frac{\partial N_{27}}{\partial \beta} = -2 * (1 - \alpha * \alpha) * \beta * (1 - \gamma * \gamma)$$

$$\frac{\partial N_1}{\partial \gamma} = \alpha * (\alpha - 1) * \beta * (\beta - 1) * (2 * \gamma - 1)/8$$
$$\frac{\partial N_2}{\partial \gamma} = \alpha * (\alpha + 1) * \beta * (\beta - 1) * (2 * \gamma - 1)/8$$
$$\frac{\partial N_3}{\partial \gamma} = \alpha * (\alpha + 1) * \beta * (\beta + 1) * (2 * \gamma - 1)/8$$
$$\frac{\partial N_4}{\partial \gamma} = \alpha * (\alpha - 1) * \beta * (\beta + 1) * (2 * \gamma - 1)/8$$
$$\frac{\partial N_5}{\partial \gamma} = \alpha * (\alpha - 1) * \beta * (\beta - 1) * (2 * \gamma + 1)/8$$
$$\frac{\partial N_6}{\partial \gamma} = \alpha * (\alpha + 1) * \beta * (\beta - 1) * (2 * \gamma + 1)/8$$
$$\frac{\partial N_7}{\partial \gamma} = \alpha * (\alpha + 1) * \beta * (\beta + 1) * (2 * \gamma + 1)/8$$
$$\frac{\partial N_8}{\partial \gamma} = \alpha * (\alpha - 1) * \beta * (\beta + 1) * (2 * \gamma + 1)/8$$
$$\frac{\partial N_9}{\partial \gamma} = (1 - \alpha * \alpha) * \beta * (\beta - 1) * (2 * \gamma - 1)/4$$
$$\frac{\partial N_{10}}{\partial \gamma} = \alpha * (\alpha + 1) * (1 - \beta * \beta) * (2 * \gamma - 1)/4$$
$$\frac{\partial N_{11}}{\partial \gamma} = (1 - \alpha * \alpha) * \beta * (\beta + 1) * (2 * \gamma - 1)/4$$
$$\frac{\partial N_{12}}{\partial \gamma} = \alpha * (\alpha - 1) * (1 - \beta * \beta) * (2 * \gamma - 1)/4$$
$$\frac{\partial N_{13}}{\partial \gamma} = (1 - \alpha * \alpha) * \beta * (\beta - 1) * (2 * \gamma + 1)/4$$
$$\frac{\partial N_{14}}{\partial \gamma} = \alpha * (\alpha + 1) * (1 - \beta * \beta) * (2 * \gamma + 1)/4$$
$$\frac{\partial N_{15}}{\partial \gamma} = (1 - \alpha * \alpha) * \beta * (\beta + 1) * (2 * \gamma + 1)/4$$
$$\frac{\partial N_{16}}{\partial \gamma} = \alpha * (\alpha - 1) * (1 - \beta * \beta) * (2 * \gamma + 1)/4$$
$$\frac{\partial N_{17}}{\partial \gamma} = -\alpha * (\alpha - 1) * \beta * (\beta - 1) * \gamma/2$$
$$\frac{\partial N_{18}}{\partial \gamma} = -\alpha * (\alpha + 1) * \beta * (\beta - 1) * \gamma/2$$
$$\frac{\partial N_{19}}{\partial \gamma} = -\alpha * (\alpha + 1) * \beta * (\beta + 1) * \gamma/2$$
$$\frac{\partial N_{20}}{\partial \gamma} = -\alpha * (\alpha - 1) * \beta * (\beta + 1) * \gamma/2$$
$$\frac{\partial N_{21}}{\partial \gamma} = -\alpha * (\alpha - 1) * (1 - \beta * \beta) * \gamma$$
$$\frac{\partial N_{22}}{\partial \gamma} = -\alpha * (\alpha + 1) * (1 - \beta * \beta) * \gamma$$
$$\frac{\partial N_{23}}{\partial \gamma} = -(1 - \alpha * \alpha) * \beta * (\beta - 1) * \gamma$$
$$\frac{\partial N_{24}}{\partial \gamma} = -(1 - \alpha * \alpha) * \beta * (\beta + 1) * \gamma$$
$$\frac{\partial N_{25}}{\partial \gamma} = (1 - \alpha * \alpha) * (1 - \beta * \beta) * (2 * \gamma - 1)/2$$
$$\frac{\partial N_{26}}{\partial \gamma} = (1 - \alpha * \alpha) * (1 - \beta * \beta) * (2 * \gamma + 1)/2$$
$$\frac{\partial N_{27}}{\partial \gamma} = -2 * (1 - \alpha * \alpha) * (1 - \beta * \beta) * \gamma$$

Linear 6 Node Wedge



$$-1 \le \alpha \le 1$$
$$0 \le \beta, \gamma \le 1$$
$$\beta + \gamma \le 1$$

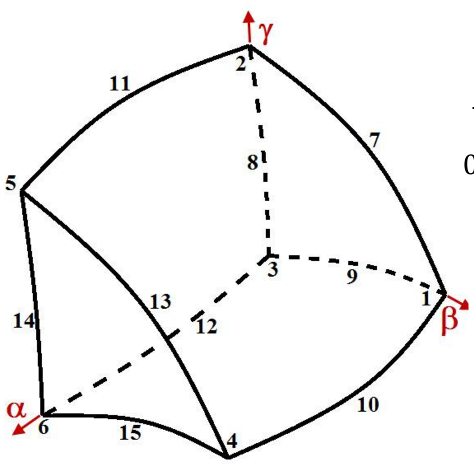
$N_1 = \beta * (1-\alpha)/2$
$N_2 = \gamma * (1-\alpha)/2$
$N_3 = (1-\beta-\gamma) * (1-\alpha)/2$
$N_4 = \beta * (1+\alpha)/2$
$N_5 = \gamma * (1+\alpha)/2$
$N_6 = (1-\beta-\gamma) * (1+\alpha)/2$

$\frac{\partial N_1}{\partial \alpha} = -\beta/2$
$\frac{\partial N_2}{\partial \alpha} = -\gamma/2$
$\frac{\partial N_3}{\partial \alpha} = -(1-\beta-\gamma)/2$
$\frac{\partial N_4}{\partial \alpha} = \beta/2$
$\frac{\partial N_5}{\partial \alpha} = \gamma/2$
$\frac{\partial N_6}{\partial \alpha} = (1-\beta-\gamma)/2$

$\frac{\partial N_1}{\partial \beta} = (1-\alpha)/2$
$\frac{\partial N_2}{\partial \beta} = 0$
$\frac{\partial N_3}{\partial \beta} = -(1-\alpha)/2$
$\frac{\partial N_4}{\partial \beta} = (\alpha+1)/2$
$\frac{\partial N_5}{\partial \beta} = 0$
$\frac{\partial N_6}{\partial \beta} = -(\alpha+1)/2$

$\frac{\partial N_1}{\partial \gamma} = 0$
$\frac{\partial N_2}{\partial \gamma} = (1-\alpha)/2$
$\frac{\partial N_3}{\partial \gamma} = (\alpha-1)/2$
$\frac{\partial N_4}{\partial \gamma} = 0$
$\frac{\partial N_5}{\partial \gamma} = (\alpha+1)/2$
$\frac{\partial N_6}{\partial \gamma} = -(\alpha+1)/2$

Quadratic 15 Node Wedge



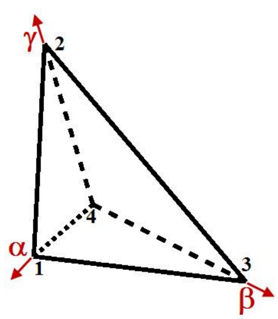
$$-1 \le \alpha \le 1$$
$$0 \le \beta, \gamma \le 1$$
$$\beta + \gamma \le 1$$

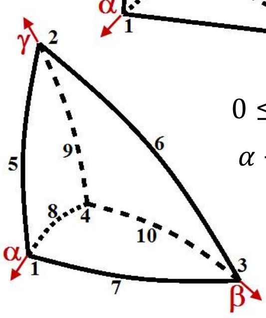
$N_1 = (1-\alpha) * \beta * (2*\beta - 2 - \alpha)/2$
$N_2 = (1-\alpha) * \gamma * (2*\gamma - 2 - \alpha)/2$
$N_3 = (\alpha - 1) * (1-\beta-\gamma) * (\alpha + 2*\beta + 2*\gamma)/2$
$N_4 = (1+\alpha) * \beta * (2*\beta - 2 + \alpha)/2$
$N_5 = (1+\alpha) * \gamma * (2*\gamma - 2 + \alpha)/2$
$N_6 = -(\alpha+1) * (1-\beta-\gamma) * (-\alpha + 2*\beta + 2*\gamma)/2$
$N_7 = 2 * (1-\alpha) * \beta * \gamma$
$N_8 = 2 * (1-\alpha) * \gamma * (1-\beta-\gamma)$
$N_9 = 2 * (1-\alpha) * \beta * (1-\beta-\gamma)$
$N_{10} = (1 - \alpha*\alpha) * \beta$
$N_{11} = (1 - \alpha*\alpha) * \gamma$
$N_{12} = (1 - \alpha*\alpha) * (1-\beta-\gamma)$
$N_{13} = 2 * (1+\alpha) * \beta * \gamma$
$N_{14} = 2 * (1+\alpha) * \gamma * (1-\beta-\gamma)$
$N_{15} = 2 * (1+\alpha) * b * (1-\beta-\gamma)$

$\frac{\partial N_1}{\partial \alpha} = \beta * (-2*\beta + 2*\alpha + 1)/2$
$\frac{\partial N_2}{\partial \alpha} = \gamma * (-2*\gamma + 2*\alpha + 1)/2$
$\frac{\partial N_3}{\partial \alpha} = (1-\beta-\gamma) * (2*\alpha + 2*\beta + 2*\gamma - 1)/2$
$\frac{\partial N_4}{\partial \alpha} = \beta * (2*\beta - 1 + 2*\alpha)/2$
$\frac{\partial N_5}{\partial \alpha} = \gamma * (2*\gamma - 1 + 2*\alpha)/2$
$\frac{\partial N_6}{\partial \alpha} = (1-\beta-\gamma) * (2*\alpha - 2*\beta - 2*\gamma + 1)/2$
$\frac{\partial N_7}{\partial \alpha} = -2 * \beta * \gamma$
$\frac{\partial N_8}{\partial \alpha} = -2 * \gamma * (1-\beta-\gamma)$
$\frac{\partial N_9}{\partial \alpha} = -2 * \beta * (1-\beta-\gamma)$
$\frac{\partial N_{10}}{\partial \alpha} = -2 * \alpha * \beta$
$\frac{\partial N_{11}}{\partial \alpha} = -2 * \alpha * \gamma$
$\frac{\partial N_{12}}{\partial \alpha} = -2 * \alpha * (1-\beta-\gamma)$
$\frac{\partial N_{13}}{\partial \alpha} = 2 * \beta * \gamma$
$\frac{\partial N_{14}}{\partial \alpha} = 2 * c * (1-\beta-\gamma)$
$\frac{\partial N_{15}}{\partial \alpha} = 2 * \beta * (1-\beta-\gamma)$

$$\frac{\partial N_1}{\partial \beta} = (1-\alpha)*(4*\beta-2-\alpha)/2$$
$$\frac{\partial N_2}{\partial \beta} = 0$$
$$\frac{\partial N_3}{\partial \beta} = (\alpha-1)*(-\alpha-4*\beta-4*\gamma+2)/2$$
$$\frac{\partial N_4}{\partial \beta} = (1+\alpha)*(4*\beta-2+\alpha)/2$$
$$\frac{\partial N_5}{\partial \beta} = 0$$
$$\frac{\partial N_6}{\partial \beta} = (\alpha+1)*(-\alpha+4*\beta+4*\gamma-2)/2$$
$$\frac{\partial N_7}{\partial \beta} = 2*\gamma*(1-\alpha)$$
$$\frac{\partial N_8}{\partial \beta} = -2*\gamma*(1-\alpha)$$
$$\frac{\partial N_9}{\partial \beta} = 2*(1-\alpha)*(1-2*\beta-\gamma)$$
$$\frac{\partial N_{10}}{\partial \beta} = 1-\alpha*\alpha$$
$$\frac{\partial N_{11}}{\partial \beta} = 0$$
$$\frac{\partial N_{12}}{\partial \beta} = \alpha*\alpha-1$$
$$\frac{\partial N_{13}}{\partial \beta} = 2*\gamma*(1+\alpha)$$
$$\frac{\partial N_{14}}{\partial \beta} = -2*\gamma*(1+\alpha)$$
$$\frac{\partial N_{15}}{\partial \beta} = 2*(1+\alpha)*(1-2*\beta-\gamma)$$

$$\frac{\partial N_1}{\partial \gamma} = 0$$
$$\frac{\partial N_2}{\partial \gamma} = (1-\alpha)*(4*\gamma-2-\alpha)/2$$
$$\frac{\partial N_3}{\partial \gamma} = (\alpha-1)*(-\alpha-4*\beta-4*\gamma+2)/2$$
$$\frac{\partial N_4}{\partial \gamma} = 0$$
$$\frac{\partial N_5}{\partial \gamma} = (1+\alpha)*(4*\gamma-2+\alpha)/2$$
$$\frac{\partial N_6}{\partial \gamma} = (\alpha+1)*(-\alpha+4*\beta+4*\gamma-2)/2$$
$$\frac{\partial N_7}{\partial \gamma} = 2*\beta*(1-\alpha)$$
$$\frac{\partial N_8}{\partial \gamma} = 2*(1-\alpha)*(1-\beta-2*\gamma)$$
$$\frac{\partial N_9}{\partial \gamma} = -2*\beta*(1-\alpha)$$
$$\frac{\partial N_{10}}{\partial \gamma} = 0$$
$$\frac{\partial N_{11}}{\partial \gamma} = 1-\alpha*\alpha$$
$$\frac{\partial N_{12}}{\partial \gamma} = \alpha*\alpha-1$$
$$\frac{\partial N_{13}}{\partial \gamma} = 2*\beta*(1+\alpha)$$
$$\frac{\partial N_{14}}{\partial \gamma} = 2*(1+\alpha)*(1-\beta-2*\gamma)$$
$$\frac{\partial N_{15}}{\partial \gamma} = -2*\beta*(1+\alpha)$$





$$0 \le \alpha, \beta, \gamma \le 1$$
$$\alpha+\beta+\gamma \le 1$$

$N_1 = \beta*(2*\beta-1)$
$N_2 = \gamma*(2*\gamma-1)$
$N_3 = (1-\alpha-\beta-\gamma)*(1-2*\alpha-2*\beta-2*\gamma)$
$N_4 = \alpha*(2*\alpha-1)$
$N_5 = 4*\beta*\gamma$
$N_6 = 4*\gamma*(1-\alpha-\beta-\gamma)$
$N_7 = 4*\beta*(1-\alpha-\beta-\gamma)$
$N_8 = 4*\alpha*\beta$
$N_9 = 4*\alpha*\gamma$
$N_{10} = 4*\alpha*(1-\alpha-\beta-\gamma)$

Linear 4 Node Tetrahedron

$N1 = a$
$N2 = g$
$N3 = b$
$N4 = 1-a-b-g$

$$\frac{\partial N_1}{\partial \alpha} = 1 \quad \frac{\partial N_2}{\partial \alpha} = 0 \quad \frac{\partial N_3}{\partial \alpha} = 0 \quad \frac{\partial N_4}{\partial \alpha} = -1$$
$$\frac{\partial N_1}{\partial \beta} = 0 \quad \frac{\partial N_2}{\partial \beta} = 0 \quad \frac{\partial N_3}{\partial \beta} = 1 \quad \frac{\partial N_4}{\partial \beta} = -1$$
$$\frac{\partial N_1}{\partial \gamma} = 0 \quad \frac{\partial N_2}{\partial \gamma} = 1 \quad \frac{\partial N_3}{\partial \gamma} = 0 \quad \frac{\partial N_4}{\partial v} = -1$$

Quadratic 10 Node Tetrahedron

$$\frac{\partial N_1}{\partial \alpha} = 0$$
$$\frac{\partial N_2}{\partial \alpha} = 0$$
$$\frac{\partial N_3}{\partial \alpha} = 4*(\alpha+\beta+\gamma)-3$$
$$\frac{\partial N_4}{\partial \alpha} = 4*\alpha-1$$
$$\frac{\partial N_5}{\partial \alpha} = 0$$
$$\frac{\partial N_6}{\partial \alpha} = -4*\gamma$$
$$\frac{\partial N_7}{\partial \alpha} = -4*\beta$$
$$\frac{\partial N_8}{\partial \alpha} = 4*\beta$$
$$\frac{\partial N_9}{\partial \alpha} = 4*\gamma$$
$$\frac{\partial N_{10}}{\partial \alpha} = 4*(1-2*\alpha-\beta-\gamma)$$

$$\frac{\partial N_1}{\partial \beta} = 4*\beta-1$$
$$\frac{\partial N_2}{\partial \beta} = 0$$
$$\frac{\partial N_3}{\partial \beta} = 4*(\alpha+\beta+\gamma)-3$$
$$\frac{\partial N_4}{\partial \beta} = 0$$
$$\frac{\partial N_5}{\partial \beta} = 4*\beta$$
$$\frac{\partial N_6}{\partial \beta} = -4*\gamma$$
$$\frac{\partial N_7}{\partial \beta} = 4*(1-\alpha-2*\beta-\gamma)$$
$$\frac{\partial N_8}{\partial \beta} = 4*\alpha$$
$$\frac{\partial N_9}{\partial \beta} = 0$$
$$\frac{\partial N_{10}}{\partial \beta} = -4*\alpha$$

$$\frac{\partial N_1}{\partial \gamma} = 0$$
$$\frac{\partial N_2}{\partial \gamma} = 4*\gamma-1$$
$$\frac{\partial N_3}{\partial \gamma} = 4*(\alpha+\beta+\gamma)-3$$
$$\frac{\partial N_4}{\partial \gamma} = 0$$
$$\frac{\partial N_5}{\partial \gamma} = 4*\beta$$
$$\frac{\partial N_6}{\partial \gamma} = 4*(1-\alpha-\beta-2*\gamma)$$
$$\frac{\partial N_7}{\partial \gamma} = -4*\beta$$
$$\frac{\partial N_8}{\partial \gamma} = 0$$
$$\frac{\partial N_9}{\partial \gamma} = 4*\alpha$$
$$\frac{\partial N_{10}}{\partial \gamma} = -4*\alpha$$